Càlcul d'integrals indefinides

Siga $f:[a,b] \to R$

S'anomena primitiva de $f(x)$ a tota funció $F(x)$ derivable en $[a,b]$ tal que $F'(x) = f(x)$.

S'anomena integral indefinida de la funció al conjunt $F(x) + C$ on $F(x)$ és una primitiva de $f(x)$. Ho denotarem per:

$\int f(x)\, dx$.

Propietats de les integrals indefinides:

1.- Si $\alpha \in R$, $\int \alpha f(x)\, dx = \alpha \int f(x)\, dx$

2.- $\int (f+g)(x)\, dx = \int f(x)\, dx + \int g(x)\, dx$.

Integrals immediates

Tipus	Formes	
	Simples	Compostes
Potencial $n \neq -1$	$\int k\, dx = kx + C$ $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$	$\int f^n \cdot f'\, dx = \frac{f^{n+1}}{n+1} + C$
Logarítmic	$\int \frac{1}{x} dx = \ln\lvert x\rvert + C$	$\int \frac{f'}{f} dx = \ln\lvert f\rvert + C$
Exponencial $a > 0, a \neq 1$	$\int e^x\, dx = e^x + C$ $\int a^x\, dx = \frac{a^x}{\ln a} + C$	$\int e^f \cdot f'\, dx = e^f + C$ $\int a^f \cdot f'\, dx = \frac{a^f}{\ln a} + C$
Cosinus	$\int \sin x\, dx = -\cos x + C$	$\int (\sin f) f'\, dx = -\cos f + C$
Sinus	$\int \cos x\, dx = \sin x + C$	$\int (\cos f) f'\, dx = \sin f + C$
Tangent	$\int (1 + tg^2 x) dx = tg\, x + C$ $\int \frac{dx}{\cos^2 x} = tg\, x + C$	$\int (1 + tg^2 f) f'\, dx = tg\, f + C$ $\int \frac{f'}{\cos^2 f} dx = tg\, f + C$
Cotangent	$\int (1 + ctg^2 x) dx = -ctg\, x + C$ $\int \frac{dx}{\sin^2 x} = -ctg\, x + C$	$\int (1 + ctg^2 f) f'\, dx = -ctg\, f + C$ $\int \frac{f'}{\sin^2 f} dx = -ctg\, f + C$
Arc sinus	$\int \frac{1}{\sqrt{1 - x^2}} dx = \arcsin x + C$	$\int \frac{f'}{\sqrt{1 - f^2}} dx = \arcsin f + C$
Arc tangent	$\int \frac{1}{1 + x^2} dx = arctg x + C$	$\int \frac{f'}{1 + f^2} dx = arctg f + C$

Tipus potencial

$\int x^4 \, dx$

$\int 3x^5 \, dx$

$\int (2x^3 - 4x^2 + 1) dx$

$\int \frac{1}{x^3} dx$

$\int \sqrt[3]{x^2} \, dx$

$\int (x+1)^2 \, dx$

$\int (2x+3)^5 dx$

$\int (x^2 + x + 1)^4 (2x+1) \, dx$

$\int \sin^3 x \cdot \cos x \, dx$

$\int \frac{\ln x}{x} dx$

Tipus logarítmic

$\int \frac{3}{x} dx$

$\int \frac{-4}{5x} dx$

$\int \frac{3x^2 + 1}{x^3 + x + 5} dx$

$\int \frac{x}{1 + x^2} dx$

$\int \frac{1}{ax + b} dx$

$\int \frac{1}{x \cdot \ln x} dx$

$\int \text{tg}\, x \, dx$

Tipus exponencial

$$\int 3^x \, dx$$

$$\int \frac{3^x}{2^x} dx$$

$$\int 5xe^{x^2} dx$$

$$\int e^{\sin x} \cos x \, dx$$

$$\int 8^{2x+1} dx$$

$$\int \frac{4^{\text{arctgx}}}{1+x^2} dx$$

Tipus sinus-cosinus

$$\int 3\cos x \, dx$$

$$\int -5\sin x \, dx$$

$$\int \frac{\sin 3x}{8} dx$$

$$\int \cos(5x+1) \, dx$$

$$\int e^x \sin\left(e^x\right) dx$$

$$\int x^3 \cos\left(2x^4+1\right) dx$$

Tipus tangent-cotangent

$$\int \frac{8}{\sin^2 x} dx$$

$$\int \left(5+5\text{tg}^2 x\right) dx$$

$$\int \frac{x}{\cos^2 x^2} dx$$

$$\int \text{tg}^2 x \, dx$$

$$\int \text{tg}^2 3x \, dx$$

Tipus arc sinus

$$\int \frac{4}{\sqrt{1-x^2}}\,dx$$

$$\int \frac{x}{\sqrt{1-x^4}}\,dx$$

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$$

$$\int \frac{1}{x\sqrt{1-\ln^2 x}}\,dx$$

$$\int \frac{1}{\sqrt{x}\sqrt{1-x}}\,dx$$

$$\int \frac{x^2}{\sqrt{1-x^6}}\,dx$$

Tipus arc tangent

$$\int \frac{1}{3+3x^2}\,dx$$

$$\int \frac{1}{1+9x^2}\,dx$$

$$\int \frac{x}{1+x^4}\,dx$$

$$\int \frac{\cos x}{1+\sin^2 x}\,dx$$

$$\int \frac{e^x}{1+e^{2x}}\,dx$$

$$\int \frac{1}{a^2+x^2}\,dx$$

$$\int \frac{x^2}{1+4x^6}\,dx$$

Integrals per substitució:

Moltes vegades fent el canvi $x = \theta(t)$ la integral es transforma en una d'altra més senzilla.
Aleshores,
$dx = \theta'(t)\,dt$

$\int f(x)\,dx = \int f(\theta(t))\cdot\theta'(t)\,dt$

Al final es desfà el canvi substituint t per $\theta^{-1}(x)$.

Problema 1:
Calculeu $\int x^2\sqrt{x-7}\,dx$

Solució:
Efectuem el canvi $t = \sqrt{x-7}$, $x = 7 + t^2$
$dx = 2t\,dt$.

$$\int x^2\sqrt{x-7}\,dx = \int \left(7+t^2\right)^2 \cdot t\cdot 2t\,dt = \int \left(2t^6 + 28t^4 + 98t^2\right)dt =$$

$$= \frac{2}{7}t^7 + \frac{28}{5}t^5 + \frac{98}{3}t^3 + C =$$

$$= \frac{2}{7}\sqrt{(x-7)^7} + \frac{28}{5}\sqrt{(x-7)^5} + \frac{98}{3}\sqrt{(x-7)^3} + C .$$

Problema 2:
Calculeu $\int\sqrt{16-x^2}\,dx$

Solució:
Efectuem el canvi, $x = 4\sin t$ a fi que desaparega l'arrel.

$dx = 4\cos t\,dt$, $t = \arcsin\left(\frac{x}{4}\right)$.

$$\int\sqrt{16-x^2}\,dx = \int\sqrt{16-(4\sin t)^2}\,4\cos t\cdot dt =$$

$$= \int\sqrt{16(1-\sin^2 t)}\,4\cos t\,dt =$$

$$= \int 16\cos^2 t\,dt =$$

Recordem que $\cos^2 t = \frac{1}{2}\left(1+\cos 2t\right)$

$$= \int 8\,\frac{1}{2}(1+\cos 2t)\,dt =$$

$$= 8\left(\int dt + \frac{1}{2}\int 2\cos 2t)\,dt\right) =$$

$$= 8t + 4\sin 2t + C =$$

$$= 8\arcsin\left(\frac{x}{4}\right) + 4\sin\left(2\arcsin\left(\frac{x}{4}\right)\right) + C .$$

Exercicis Integrals per substitució:

$$\int \frac{1}{(1+x)\sqrt{x}}\,dx$$

$$\int \frac{x}{\sqrt{1+x}}\,dx$$

$$\int x\sqrt{1+x}\,dx$$

$$\int \sqrt{e^x-1}\,dx$$

$$\int x\sqrt{3+2x}\,dx$$

$$\int \frac{e^x}{e^x+1}\,dx$$

$$\int \frac{e^{2x}}{e^x+1}\,dx$$

$$\int \frac{e^x}{e^{2x}+1}\,dx$$

Integració per parts

$d(uv) = d(u)\cdot v + u\cdot d(v)$

Aleshores,

$uv = \int d(uv) = \int v\cdot d(u) + \int u\cdot d(v)$

Per tant,

$\int u\cdot d(v) = uv - \int v\cdot d(u)$

Aleshores el mètode s'aplicarà sempre que $\int v\cdot d(u)$ tinga menys dificultat de càlcul integral que la integral de partida i que $d(v)$ siga fàcilment integrable.

Problema:

Calculeu $\int xe^x\ dx$.

Solució:

Fent el canvi $u = x$, $d(u) = 1\cdot dx$

$d(v) = e^x\ dx$, $v = \int e^x\ dx = e^x$.

Aplicant el mètode:

$\int xe^x\ dx = xe^x - \int e^x\ dx = xe^x - e^x + C = (x-1)e^x + C$.

Problema:

Calculeu $I = \int e^x \sin x\ dx$.

Solució:

Fent el canvi $u = e^x$ $d(u) = e^x\ dx$

$d(v) = \sin x\ dx$, $v = \int \sin x\ dx = -\cos x$.

Aplicant el mètode:

$I = \int e^x \sin x\ dx = -\cos x\cdot e^x - \int(-\cos x)e^x\ dx = -\cos x\cdot e^x + \int \cos x\cdot e^x\ dx$

Fem el canvi $u = e^x$ $d(u) = e^x\ dx$

$d(v) = \cos x\ dx$, $v = \int \cos x\ dx = \sin x$.

Per tant, aplicant el mètode:

$\int \cos x\cdot e^x\ dx = e^x\cdot \sin x - \int e^x \sin x\ dx$.

Aleshores,

$I = \int e^x \sin x\ dx = -\cos x\cdot e^x + \int \cos x\cdot e^x\ dx = -\cos x\cdot e^x + \sin x\cdot e^x - I$

Resolem l'equació en la incògnita I:

$2I = -\cos x\cdot e^x + \sin x\cdot e^x$

Aleshores,

$$I = \frac{e^x(\sin x - \cos x)}{2} + C.$$

Exercicis integració per parts:

$\int \ln x \, dx$

$\int x \cdot \cos x \, dx$

$\int x^2 e^x \, dx$

$\int x \cdot \ln x \, dx$

$\int e^x \cos x \, dx$

$\int \text{arctg} x \, dx$

$\int e^{2x} \sin x \, dx$

$\int e^x \cos 3x \, dx$

$\int \sin^4 x \, dx$

$\int x \cdot 7^x \, dx$

$\int \frac{\ln x}{x^3} dx$

Integració de funcions racionals

Siga $\int \frac{p(x)}{q(x)}dx$, tal que $p(x), q(x)$ són polinomis.

Es pot suposar que el grau de $p(x)$ es menor que el grau de $q(x)$.
Si el grau de $p(x)$ és major o igual que el grau de $q(x)$, realitzaríem prèviament la divisió:
$\frac{p(x)}{q(x)} = h(x) + \frac{r(x)}{q(x)}$, tal que $h(x)$, $r(x)$ són polinomis i grau de $r(x)$ es menor que el grau de $q(x)$.

Siga $p(x)$, $q(x)$ dos polinomis tal que el grau de $p(x)$ és major o igual que el grau de $q(x)$.
Si coneixem les arrels del denominador $q(x)$ és poden plantejar 4 situacions:

a) Arrels reals simples.
b) Arrels reals múltiples.
c) Arrels complexes Simples
d) Arrels complexes múltiples (Aquest cas no l'estudiarem).

a) Arrels reals simples
Si les arrels de $q(x)$ són $x_1, x_2, \ldots\ldots, x_n$ reals i distintes.
$q(x) = (x - x_1)(x - x_2) \cdot \ldots\ldots \cdot (x - x_n)$.
Aleshores,:
$\frac{p(x)}{q(x)} = \frac{A_1}{x - x_1} + \frac{A_2}{x - x_2} + \ldots\ldots + \frac{A_n}{x - x_n}$ on $A_1, A_2, \ldots\ldots, A_n \in R$.
Per tant:
$$\int \frac{p(x)}{q(x)}dx = \int \frac{A_1}{x - x_1}dx + \int \frac{A_2}{x - x_2}dx + \ldots\ldots + \int \frac{A_n}{x - x_n}dx =$$
$$= A_1 \ln|x - x_1| + A_2 \ln|x - x_2| + \ldots\ldots + A_n \ln|x - x_2| + C$$

Problema
Calculeu $\int \frac{1}{x^2 - 2x - 3}dx$

Solució:
El grau del numerador és menor que el grau del denominador.
Les arrels del denominador són $x = -1,3$
$x^2 - 2x - 3 = (x + 1)(x - 3)$.
$\frac{1}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}$. Hem de determinar $A, B \in R$
$1 = A(x - 3) + B(x + 1)$.
$1 = (A + B)x - 3A + B$.
Identificant coeficients dels polinomis ens queda el següent sistema:
$\begin{cases} A + B = 0 \\ -3A + B = 1 \end{cases}$, la solució del qual és, $\begin{cases} A = -\frac{1}{4} \\ B = \frac{1}{4} \end{cases}$ Aleshores:

$$\int \frac{1}{x^2-2x-3}dx = \frac{-1}{4}\int \frac{1}{x+1}dx + \frac{1}{4}\int \frac{1}{x-3}dx = \frac{-1}{4}\ln|x+1| + \frac{1}{4}\ln|x-3| + C.$$

Problema:

Calculeu $\int \frac{x^3+9x^2+23x+17}{x^3+6x^2+11x+6}dx$

Solució:

El grau del numerador no és menor que el grau del denominador. Efectuem la divisió:

$$\frac{x^3+9x^2+23x+17}{x^3+6x^2+11x+6} = 1 + \frac{3x^2+12x+11}{x^3+6x^2+11x+6}$$

Calculem les arrels del denominador amb la regla de Ruffini:

$x=-1,-2,-3$, aleshores:

$x^3+6x^2+11x+6=(x+1)(x+2(x+3)$

$\frac{3x^2+12x+11}{x^3+6x^2+11x+6} = \frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{x+3}$. Hem de determinar $A,B,C \in R$.

$3x^2+12x+11 = A(x+2)(x+3)+B(x+1)(x+3)+C(x+1)(x+2)$

Per a $x=-1$,

$3(-1)^2+12(-1)+11 = A(-1+2)(-1+3)+B(-1+1)(-1+3)+C(-1+1)(-1+2)$

$2=2A$, Per tant, $A=1$.

Per a $x=-2$, $-1=-B$, Per tant, $B=1$.

Per a $x=-3$, $2=2C$, Per tant, $C=1$.

Aleshores,

$$\int \frac{x^3+9x^2+23x+17}{x^3+6x^2+11x+6}dx = \int \left(1+\frac{3x^2+12x+11}{x^3+6x^2+11x+6}\right)dx =$$

$$= \int \left(x+\frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}\right)dx =$$

$$= \frac{x^2}{2}+\ln|x+1|+\ln|x+2|+\ln|x+3|+C$$

Exercicis d’integrals racionals arrels reals simples

$$\int \frac{x+1}{x^2-2x}dx$$

$$\int \frac{x^3+1}{x^2-4}dx$$

$$\int \frac{1}{x(x+2)(x-1)}dx$$

$$\int \frac{x^2+1}{x^3-7x-6}dx$$

b) Arrels reals múltiples

Si les arrels de $q(x)$ són $x_1, x_2, \ldots, x_n$ reals de multiplicitat $r_1, r_2, \ldots, r_n$, respectivament.

$q(x) = (x - x_1)^{r_1}(x - x_2)^{r_2} \cdot \ldots \cdot (x - x_n)^{r_n}$.

Aleshores:

$$\frac{p(x)}{q(x)} = \frac{A_{11}}{x - x_1} + \frac{A_{12}}{(x - x_1)^2} + \ldots + \frac{A_{1r_1}}{(x - x_1)^{r_1}} + \ldots + \frac{A_{n1}}{x - x_n} + \frac{A_{n2}}{(x - x_n)^2} + \ldots + \frac{A_{nr_n}}{(x - x_n)^{r_n}}$$ on

$A_{ij} \in R$.

Per tant:

$$\int \frac{p(x)}{q(x)}dx = \int \frac{A_{11}}{x - x_1}dx + \int \frac{A_{12}}{(x - x_1)^2}dx + \ldots + \int \frac{A_{1r_1}}{(x - x_1)^{r_1}}dx + \ldots +$$

$$+ \ldots + \int \frac{A_{n1}}{x - x_n}dx + \int \frac{A_{n2}}{(x - x_n)^2}dx + \ldots + \int \frac{A_{nr_n}}{(x - x_n)^{r_n}}dx =$$

$$= A_{11}\ln|x - x_1| + A_{12}\frac{-1}{x - x_1} + \ldots + A_{1r_1}\frac{1}{(1 - r_1)(x - x_1)^{r_1 - 1}} + \ldots +$$

$$+ \ldots + A_{n1}\ln|x - x_n| + A_{n2}\frac{-1}{x - x_n} + \ldots + A_{nr_1}\frac{1}{(1 - r_n)(x - x_n)^{r_n - 1}} + C$$

Problema:

Calculeu $\int \frac{3x + 1}{(x + 1)^2}dx$

Solució:

El grau del numerador és menor que el grau del denominador.

L'arrel del denominador són $x = -1$ de multiplicitat 2.

$\frac{3x + 1}{(x + 1)^2} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2}$. Hem de determinar $A, B \in R$.

$3x + 1 = A(x + 1) + B$

$3x + 1 = Ax + A + B$. Igualant els coeficients dels polinomis:

$\begin{cases} A = 3 \\ A + B = 1 \end{cases}$, la solució del sistema és: $\begin{cases} A = 3 \\ B = -2 \end{cases}$.

Aleshores,

$$\int \frac{3x + 1}{(x + 1)^2}dx = \int \frac{3}{x + 1}dx + \int \frac{-2}{(x + 1)^2}dx = 2\ln|x + 1| + \frac{2}{x - 1} + C.$$

Exercicis d'integrals racionals arrels reals múltiples

$$\int \frac{x}{x^2 - 4x + 4}dx$$

$$\int \frac{1}{x^4 + 4x^3 + 4x^2}dx$$

$$\int \frac{x + 1}{(x^2 - 4)^2}dx$$

$$\int \frac{2x+1}{x^3-6x^2+9x}dx$$

$$\int \frac{1}{(x+1)^2(x-3)^2}dx$$

c) Arrels complexes simples

Exemple

Calculeu $\int \frac{4x+1}{x^2+2x+5}dx$.

$x^2+2x+5=0$ no té solució real. Té dues arrels complexes distintes.
Hem de aconseguir que la integral es transforme en dues integrals immediates: una de tipus logarítmic i l'altra de tipis arc tangent.

La derivada del denominador és $2x+2$.
El numerador és: $4x+1=2(2x+2)-3$.

$$\int \frac{4x+1}{x^2+2x+5}dx = 2\int \frac{2x+2}{x^2+2x+5}dx - 3\int \frac{1}{x^2+2x+5}dx.$$

La primera és immediata de tipus logaritme.
Els denominador completant quadrats és: $x^2+2x+5=(x+1)^2+4$.

$$\int \frac{4x+1}{x^2+2x+5}dx = 2\ln\left|x^2+2x+5\right| - 3\int \frac{1}{(x+1)^2+4}dx.$$

Dividim numerador i denominador de la segona integral per 4:

$$\int \frac{4x+1}{x^2+2x+5}dx = 2\ln\left|x^2+2x+5\right| - 3\int \frac{\frac{1}{4}}{\left(\frac{x+1}{2}\right)^2+1}dx.$$

$$\int \frac{4x+1}{x^2+2x+5}dx = 2\ln\left|x^2+2x+5\right| - 3\frac{1}{4}2\int \frac{\frac{1}{2}}{\left(\frac{x+1}{2}\right)^2+1}dx = 2\ln\left|x^2+2x+5\right| + \frac{3}{2}\text{arctg}\left(\frac{x+1}{2}\right)+C$$

Si les arrels de $q(x)$ són complexes simples.

$$q(x) = \left((x-a_1)^2+b_1^2\right)\left((x-a_2)^2+b_2^2\right)\ldots\left((x-a_n)^2+b_n^2\right).$$

$$\int \frac{p(x)}{q(x)}dx = \int \frac{A_1x+B_1}{(x-a_1)^2+b_1^2}dx + \int \frac{A_2x+B_2}{(x-a_2)^2+b_2^2}dx + \ldots\ldots + \int \frac{A_nx+B_n}{(x-a_n)^2+b_n^2}dx.$$

Cadascuna de les anterior és transforma en una integral de tipus logarítmic i una de tipus arc tangent.

Exemple:

Calculeu $\int \frac{x^2+x+1}{x^3-4x^2+5x}dx$.

Calculem els zeros del denominador:

$x^3-4x^2+5x = x\left(x^2-4x+5\right)$. El denominador té una arrel real $x=0$ i dos complexes simples.

$$\frac{x^2+x+1}{x^3-4x^2+5x}=\frac{A}{x}+\frac{Bx+C}{x^2-4x+5}.$$

$$\frac{x^2+x+1}{x^3-4x^2+5x}=\frac{A}{x}+\frac{A(x^2-4x+5)+(Bx+C)x}{x(x^2-4x+5)}.$$

Els denominadors són iguals.
Donant valors al numerador:

Si $x=0$, $1=5A$, aleshores, $A=\frac{1}{5}$.

Si $x=1$, $3=2A+B+C$.
Si $x=-1$ $1=10A+B-C$.

$\begin{cases} B+C=\frac{13}{5} \\ B-C=-1 \end{cases}$. Resolent el sistema: $\begin{cases} B=\frac{4}{5} \\ C=\frac{9}{5} \end{cases}$.

$$\int\frac{x^2+x+1}{x^3-4x^2+5x}dx=\frac{1}{5}\int\frac{1}{x}dx+\frac{1}{5}\int\frac{4x+9}{x^2-4x+5}dx.$$

La primera integral és immediata de tipus logarítmic i la segona l'hem de transformar en dues integrals una de tipus logarítmic i una de tipus arc tangent.

$$\int\frac{x^2+x+1}{x^3-4x^2+5x}dx=\frac{1}{5}\ln|x|+\frac{1}{5}\cdot\left(2\int\frac{2x-4}{x^2-4x+5}dx+17\int\frac{1}{(x-2)^2+1}dx\right)=$$

$$=\frac{1}{5}\ln|x|+\frac{2}{5}\ln\left|x^2-4x+5\right|+\frac{17}{5}\operatorname{arctg}(x-2)+C.$$

Racionals amb canvi de variable

$$\int\frac{e^{2x}}{1+e^x}dx$$

$$\int\frac{x}{1-\sqrt{x}}dx$$

$$\int\frac{2^{2x}+1}{2^x-1}dx$$

$$\int\frac{x^2}{\sqrt{1+x}}dx$$

$$\int\frac{x}{1+\sqrt{x}}dx$$

$$\int\frac{x^3}{\sqrt{x-1}}dx$$

$$\int\frac{\sqrt{x-1}}{1-\sqrt{x-1}}dx$$

Solucions

Tipus potencial

$\int x^4\,dx = \frac{x^5}{5} + C$ $\int 3x^5\,dx = \frac{1}{2}x^6 + C$ $\int (2x^3 - 4x^2 + 1)dx$ $\int \frac{1}{x^3}dx = \frac{-1}{2x^2} + C$

$\int \sqrt[3]{x^2}\,dx = \frac{3}{5}\sqrt[3]{x^5} + C$ $\int (x+1)^2\,dx = \frac{(x+1)^3}{3} + C$ $\int (2x+3)^5 dx = \frac{1}{12}(2x+3)^6 + C$

$\int (x^2+x+1)^4(2x+1)\,dx = \frac{(x^2+x+1)^5}{5} + C$ $\int \sin^3 x \cdot \cos x\,dx = \frac{\sin^4 x}{4} + C$

$\int \frac{\ln x}{x}dx = \frac{\ln^2 x}{2} + C$

Tipus logarítmic

$\int \frac{3}{x}dx = 3\ln|x| + C$ $\int \frac{-4}{5x}dx = \frac{-4}{5}\ln|x| + C$ $\int \frac{3x^2+1}{x^3+x+5}dx = \ln|x^3+x+5| + C$

$\int \frac{x}{1+x^2}dx = \frac{1}{2}\ln|1+x^2| + C$ $\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + C$ $\int \frac{1}{x\cdot\ln x}dx = \ln|\ln|x|| + C$

$\int \mathrm{tg}\,x\,dx = -\ln|\cos x| + C$

Tipus exponencial

$\int 3^x\,dx = \frac{3^x}{\ln 3} + C$ $\int \frac{3^x}{2^x}dx = \frac{\left(\frac{3}{2}\right)^x}{\ln\frac{3}{2}} + C$ $\int 5xe^{x^2}dx = \frac{5}{2}e^{x^2} + C$

$\int e^{\sin x}\cos x\,dx = e^{\sin x} + C$ $\int 8^{2x+1}dx = \frac{1}{2}\frac{8^{2x+1}}{\ln 8} + C$ $\int \frac{4^{\mathrm{arctg}x}}{1+x^2}dx = \frac{4^{\mathrm{arctg}x}}{\ln 4} + C$

Tipus sinus-cosinus

$\int 3\cos x\,dx = 3\sin x + C$ $\int -5\sin x\,dx = 5\cos x + C$ $\int \frac{\sin 3x}{8}dx = \frac{-1}{24}\cos 3x + C$

$\int \cos(5x+1)\,dx = \frac{1}{5}\cos(5x+1) + C$ $\int e^x \sin(e^x)\,dx = -\cos(e^x) + C$

$\int x^3\cos(2x^4+1)\,dx = \frac{1}{8}\sin(2x^4+1) + C$

Tipus tangent-cotangent

$\int \frac{8}{\sin^2 x}dx = 8 - \mathrm{ctg}x + C$ $\int (5 + 5\mathrm{tg}^2 x)dx = 5\mathrm{tg}x + C$ $\int \frac{x}{\cos^2 x^2}dx = \frac{1}{2}\mathrm{tg}x^2 + C$

$\int \mathrm{tg}^2 x\,dx = -x + \mathrm{tg}x + C$ $\int \mathrm{tg}^2 3x\,dx = -x + \frac{1}{3}\mathrm{tg}(3x) + C$

Tipus arc sinus

$\int \frac{4}{\sqrt{1-x^2}}dx = 4\arcsin x + C$ $\int \frac{x}{\sqrt{1-x^4}}dx = \frac{1}{2}\arcsin x^2 + C$

$\int \frac{e^x}{\sqrt{1-e^{2x}}}dx = \arcsin(e^x) + C$ $\int \frac{1}{x\sqrt{1-\ln^2 x}}dx = \arcsin(\ln x) + C$

$\int \frac{1}{\sqrt{x}\sqrt{1-x}}dx = 2\arcsin\sqrt{x} + C$ $\int \frac{x^2}{\sqrt{1-x^6}}dx = \frac{1}{3}\arcsin x^3$

Tipus arc tangent

$\int \frac{1}{3+3x^2}\,dx = \frac{1}{3}\text{arctg}x + C$ $\quad \int \frac{1}{1+9x^2}\,dx = \frac{1}{3}\text{arctg}(3x) + C$

$\int \frac{x}{1+x^4}\,dx = \frac{1}{2}\text{arctg}x^2 + C$ $\quad \int \frac{\cos x}{1+\sin^2 x}dx = \text{arctg}(\sin x) + C$

$\int \frac{e^x}{1+e^{2x}}\,dx = \text{arctg}(e^x) + C$ $\quad \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\text{arctg}\frac{x}{a} + C$

$\int \frac{x^2}{1+4x^6}\,dx = \frac{1}{6}\text{arctg}(2x^3) + C$

Exercicis Integrals per substitució

$\int \frac{1}{(1+x)\sqrt{x}}\,dx = 2\text{arctg}\sqrt{x} + C$ $\quad \int \frac{x}{\sqrt{1+x}}\,dx = \frac{2}{3}(x+2)\sqrt{1+x} + C$

$\int x\sqrt{1+x}\,dx = \frac{2}{15}(3x-2)\sqrt{(1+x)^3} + C$ $\quad \int \sqrt{e^x-1}\,dx = 2\sqrt{e^x-1} - 2\text{arctg}\sqrt{e^x-1} + C$

$\int x\sqrt{3+2x}\,dx = \frac{-1}{5}(x+1)\sqrt{(3-2x)^3} + C$ $\quad \int \frac{e^x}{e^x+1}dx = \ln|e^x+1| + C$

$\int \frac{e^{2x}}{e^x+1}dx = e^x - \ln|e^x+1| + C$ $\quad \int \frac{e^x}{e^{2x}+1}dx = \text{arctg}(e^x) + C$

Exercicis integració per parts:

$\int \ln x\,dx = x(-1+\ln|x|) + C$ $\quad \int x\cdot\cos x\,dx = \cos x + x\cdot\sin x + C$

$\int x^2e^x\,dx = e^x(x^2-2x+2) + C$ $\quad \int x\cdot\ln x\,dx = \frac{x^2}{4}(-x^2+\ln|x|) + C$

$\int e^x\cos x\,dx = \frac{e^x}{2}(\cos x + \sin x) + C$ $\quad \int \text{arctg}x\,dx = x\cdot\text{arctg}x - \frac{1}{2}\ln|1+x^2| + C$

$\int e^{2x}\sin x\,dx = \frac{e^{2x}}{5}(2\sin x - \cos x) + C$ $\quad \int e^x\cos 3x\,dx = \frac{e^x}{10}(\cos(3x) + 3\sin(3x)) + C$

$\int \sin^4 x\,dx = \frac{3}{8}x - \frac{1}{8}\cos x(2\sin^3 x + 3\sin x) + C$ $\quad \int x\cdot 7^x\,dx = \frac{7^x}{\ln^2 7}(-1 + x\cdot\ln 7) + C$

$\int \frac{\ln x}{x^3}dx = \frac{-1}{4x^2}(1+2\ln|x|) + C$

Exercicis d'integrals racionals arrels reals simples

$\int \frac{x+1}{x^2-2x}\,dx = \frac{3}{2}\ln|x-2| - \frac{1}{2}\ln|x| + C$ $\quad \int \frac{x^3+1}{x^2-4}dx = \frac{1}{2}x^2 + \frac{7}{4}\ln|x+2| + \frac{9}{4}\ln|x-2| + C$

$\int \frac{1}{x(x+2)(x-1)}\,dx = \frac{-1}{2}\ln|x| + \frac{1}{6}\ln|x+2| + \frac{1}{3}\ln|x-1| + C$

$\int \frac{x^2+1}{x^3-7x-6}\,dx = \frac{1}{2}\ln|x-3| + \ln|x+2| + \frac{-1}{2}\ln|x+1| + C$

Exercicis d'integrals racionals arrels reals múltiples

$\int \frac{x}{x^2-4x+4}dx = \ln|x-2| + \frac{2}{x-2} + C$

$\int \frac{1}{x^4+4x^3+4x^2}\,dx = \frac{-1}{4}\ln|x| + \frac{-1}{4x} + \frac{1}{4}\ln|x+2| + \frac{-1}{4(x+2)} + C$

$\int \frac{x+1}{(x^2-4)^2}dx = \frac{1}{32}\ln|x+2| + \frac{1}{16(x+2)} - \frac{1}{32}\ln|x-2| + \frac{-3}{16(x-2)} + C$

$$\int \frac{2x+1}{x^3-6x^2+9x}dx = \frac{1}{9}\ln|x| + \frac{-1}{9}\ln|x-3| - \frac{7}{3(x-3)} + C$$

$$\int \frac{1}{(x+1)^2(x-3)^2}dx = \frac{1}{32}\ln|x+1| - \frac{1}{16(x+1)} - \frac{1}{32}\ln|x-3| - \frac{1}{16(x-3)} + C$$

Racionals amb canvi de variable

$$\int \frac{e^{2x}}{1+e^x}dx = e^x - \ln|1+e^x| + C \qquad \int \frac{x}{1-\sqrt{x}}dx = -\frac{\sqrt{x}\left(2x+\sqrt{x}+6\right)}{3} - 2\ln|1-\sqrt{x}| + C$$

$$\int \frac{2^{2x}+1}{2^x-1}dx = -x + \frac{2^x}{\ln 2} + \frac{2\ln|2^x-1|}{\ln 2} + C \qquad \int \frac{x^2}{\sqrt{1+x}}dx = \frac{2}{15}\left(3x^2-4x+8\right)\sqrt{x+1} + C$$

$$\int \frac{x}{1+\sqrt{x}}dx = -x + 2\sqrt{x} + \frac{2}{3}\sqrt{x^3} - 2\ln|1+\sqrt{x}| + C$$

$$\int \frac{x^3}{\sqrt{x-1}}dx = \frac{2}{35}\left(5x^3+6x^2+8x+16\right)\sqrt{x-1} + C$$

$$\int \frac{\sqrt{x-1}}{1-\sqrt{x-1}}dx = -x - 2\sqrt{x-1} - 2\ln\left|\sqrt{x-1}-1\right| + C$$